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Original article

Ecology Environmental Application: Mathematical Model for Wastewater Treatment Biological Process

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Abstract
In this paper, we consider a system of ecological environmental application by modeling the process of wastewater treatment. The model describes a continuously well-stirred tank containing microorganisms that degrade any degradable solid material (later referred to as the substrate). This paper discusses and illustrates the time and the concentration of the material we used, those we need to succeed this process, by studying the evolution and behavior of the system of equations to get some conditions that are necessary to treat wastewater biologically.
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Introduction

The system of non-linear ordinary differential equations is considered in this paper. This system describes the reaction inside a continuous stirred bioreactor. The bioreactor can be defined as a well-stirred tank containing microorganisms. The substrate flows at a continuous rate. The microorganisms grow in the tank through the consumption of substrate to produce more microorganisms and products (P).The rest of the microorganisms that are unused flow out of the bioreactor. Although the model is very simple, it has various applications in many fields (X)ses as a (S) model for the concentration of pollutants and microorganisms within aerated lagoons and lakes in ecological and environmental engineering [2]. Moreover, it is used to model the biological oxidation of wastewater in biochemical engineering [3]. The microorganisms grow in the tank through the consumption of substrate to produce more microorganisms and products (P).The rest of the microorganisms that are unused flow out of the bioreactor. Although the model is very simple, it has various applications in many fields. It arises as a model for the concentration of pollutants and microorganisms within aerated lagoons and lakes in ecological environmental engineering [2]. Moreover, it is used to model the biological oxidation of wastewater in biochemical engineering [3]. Moreover, it is used to model the biological oxidation of wastewater in biochemical engineering [3].

The dimensional system

$V\frac{dX}{dt} = F(X_0 - X) + VX\mu(s) - Vk_d X$ (1.1)

$V\frac{dS}{dt} = F(S_0 - S) - VX\,\frac{\mu(S)}{\alpha}$(1.2)

Residence time (The time for the reaction to take place**).**

$T = \frac{V}{F}$(1.3)

Monod's equation

$$\mu(\boldsymbol{S}) = \frac{\mu_{m(S)}}{K_s + S}. \quad (1.4)$$

The variables (S) and (X) denote the concentration of substrate and microorganisms, respectively. The initial conditions $S(0)$ and,$X(0)$, must therefore be non-negative. We denote the concentration of the substrate(S) and the microorganism$(X)$by $|S|$ and $|X|$ respectively. In this system F is the flow rate through the bioreactor?(dm^3hr^{-1}),k_s is the Monod constant $(|S|)$, S is the substrate concentration within the bioreactor$(|S|)$, S_0 is the concentration of substrate flow into the reactor $|S|$, V is the volume of the bioreactor(dm^3),X is the concentration of microorganisms within the bioreactor $(|X|)$, X_0 is the concentration of microorganisms flowing into the reactor$(|X|)$,K_d is the decay or death coefficient (hr^{-1}), t is time(hr), $\alpha$is the yield factor $|X||S|^{-1}$, μ is the specific growth rate model,μ_m is the maximum rate and specific growth ratehr^{-1} and T is the residence time(hr). For a specific wastewater, a given biological community, and aparticular set of environmental conditions, the parameters K_s, K_d, α, μ_m are fixed. The parameters that can be varied are S_0, $X_0$and$T$. In the system, the main experimental control parameter, i.e the primary bifurcation parameter, is the residence time.

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Global behavior

Nondimensionalization [4]

Fundamental ideas of scaling are to reduce the number of independent physical parameters in the model. So, we introduce dimensionless variables in the following manner:

$$S^* = \frac{S}{K_S} = \frac{|S|}{|S|} = 1 X^* = \frac{X}{\alpha_S K_S} = \frac{|X|}{|X||S||S|^{-1}} = 1 t^* = \mu_m t = (hr)(hr)^{-1} = 1$$

Where S^*, X^*, t^* are the concentration of substrate, microorganism concentrations, and time, respectively? Then we find the dimensionless model as follows:

$$\frac{dX^*}{dt^*} = \frac{1}{T^*}(X_0^* - X^*) + \frac{S^* X^*}{1+S^*} - K_d^* X^* \quad (1.5)$$

$$\frac{dS^*}{dt^*} = \frac{1}{T^*}(S_0^* - S^*) - \frac{X^* S^*}{1+S^*} \quad (1.6)$$

Where:

$S_0^* = \frac{S_0}{K_S}$, $X^*{}_0 = \frac{X_0}{\alpha K_S}$, $T^* = \frac{\mu_m V}{F}$ and $K_d^* = \frac{K_d}{\mu_m}$

In this equation S_0^* is the dimensionless substrate concentration in the feed. X_0^* is the dimensionless cell mass concentration in the feed. K_d^* is the dimensional decay coefficient and T^* is the dimensionless residence time? The main experimentally controllable parameter is dimensionless residence time. In the coming calculation, assume that, there are no microorganisms in the influent $X_0 = X_0^* = 0$ which means the growth medium fed into the bioreactor is sterile.

A positively invariant area

Definition [5]

Let $\sum$ be a domain enclosed by a simple curve $\partial \sum$ in the plane. Then $\sum$ is an invariant set for the two-component system

$$\frac{du}{dt} = f(u, v)$$

$$\frac{dv}{dt} = g(u, v)$$

If any solution of the system with the initial condition in $\sum$ remains in $\sum$ for all $t > 0$. Now, we want to show that the solution cannot escape through the various axes. Consider the region $S^* \geq 0$, and $X^* \geq 0$. Along the boundary where $S^* \geq 0$, $X^* = 0$ we find that $\frac{dX^*}{dt^*} = 0$. Therefore, the s$S^* \geq 0$ crosses this boundary., $X^* = 0$ we find that $\frac{dX^*}{dt^*} = 0$. Therefore, the solution never crosses this boundary

Along the boundary where $S^* = 0$ and $X^* > 0$ we find $\frac{dS^*}{dt^*} = \frac{S_0^*}{T^*}$. This means the solution goes into the domain; we therefore know that the region is positively invariant. It is not enough to show that the solution ina region cannot escape from the boundaries, we also need to show that, if the solution starts outside the region, it must overtime tend to it. This means we should prove the region is exponentially attractive.

Now, region R is bound by the curve $S^* + X^* = S_0^*$. let $Z = S^* + X^*$ then

$$\frac{dZ}{dt^*} = \frac{1}{T^*}(S_0^* - Z) - K_d^* X^*$$

$X^* \geq 0$ce and $(-K_d^* X^* \leq 0)$ we have the inequality $(-K_d^* X^* \leq 0)$ we have the inequality

$$\frac{dZ}{dt^*} \leq \frac{1}{dT^*}(S_0^* - Z)$$

We consider the $W(t^* = 0) = Z(t^* = 0)$ so, the values of thedifferential equations below those of the differential equation

$$\frac{dW}{dt^*} = \frac{1}{T^*}(S_0^* - W)$$

With the same initial condition. The solution of this differential equation is

$W(t^*) = S_0^* - \exp\left(-\frac{t^*}{T^*}\right)(W(0) - S_0^*)$ Then

$$S^*(t^*) + X^*(t^*) \leq S_0^* - \exp\left(-\frac{t^*}{T^*}\right)(W(0) - S_0^*)$$

We conclude that the solution of this differential equation tends to S_0^*tan exponential rate. R Therefore, an exponential attracting. R , therefore, an exponential attracting.

In case the decay rate is faster than the specific growth rate

We now show that if $K_d^* \geq \frac{S_0^*}{1+S_0^*}$ then the solutions converge t$K_d^* \geq \frac{S_0^*}{1+S_0^*}$. First, write the equation for X^* as: $(S_0^*, 0)$. First write the equation for X^* as:

$$\frac{dX^*}{dt^*} = -\left[\frac{1}{T^*} + K_d^* - \frac{S^*}{1+S^*}\right] X^*$$

We can get the maximum value of $\frac{S^*}{1+S^*}$ in the region$R$ when$S^* = S_0^*$ then

$$\frac{dX^*}{dt^*} \leq -\left[\frac{1}{T^*} + K_d^* - \frac{S_0{}^*}{1+S_0{}^*}\right] X^*$$

Therefore, if $K_d^* \geq \frac{S_0^*}{1+S_0^*}$ then$\frac{dX^*}{dt^*} \leq -\frac{1}{T^*} X^*$.Consider the function.$\frac{dZ_1^*}{dt^*} = -\frac{1}{T^*} Z_1^*$Where $Z_1(t^* = 0) = X^*(t^* = 0)$. This means that the solution$X^*(t^*)$ is below the solution of Z_1,where$X^*(t^*) \leq Z_1^*(t^*)$and it a first-order differential equation that can be solved explicitly

$$Z_1(t^*) = Z_1(0)\exp\left(-\frac{t^*}{T^*}\right)$$

Hence, the solution of the last differential equation tends to 0 at an exponential rate. Therefore, as time tends to infinity, solutions converge to the segment on the S^*axis between 0 and S_0^* .On this segment, the equation for S^* becomes:

$$\frac{dS^*}{dt^*} = \frac{1}{T^*}(S_0^* - S^*)$$

By solving this differential equation, we find that the solution tends to$(S_0^*, 0)$. This solution is known as "Washout steady-state solution". Because the steady-state value of the microorganisms' concentration is zero. As microorganisms do not flow into the reactor $X_0^* = 0$, this means that all the original microorganisms present in the system have been literally "washed out". In practical applications, we want to avoid this solution. This is because in dimensional units, this means that $K_d^* \geq \mu_m$, meaning that the decay is faster than the specific growth while in residence. Since $X_0^* = 0$ no new material enters the bioreactor, and over time microorganism decreases.

Dulac function and limit cycle of the system

Definition [6]

Dulac's function is such a function is non-negative and has the property thatρt, the divergence of $*\ vectorfield < 0$. Whena system has this property, then there can be no limit cycle, because if such a limit cycle γ were present, then the weighted area of the domain bounded by this limit cycle would decrease in time. Therefore,γcan not be an invariant closed curve. Now, we know that along the edge$X^* = 0$. If the differential equation becomes a system that has this property, then there can be no limit cycle, because γ were present, then the weighted area of the domain bounded by this limit cycle:$\int_\gamma \ \rho dxdy$decreases in time. Therefore,γcannot be an invariant closed curve. Now, we know that along the edge $X^* = 0$the differential equation becomes

$$\frac{dS^*}{dt^*} = \frac{1}{T^*}(S_0^* - S^*)$$

And its solution converges to S_0^*. Therefore, there will be no periodic solutions on this edge.

Consider the function $\rho(S^*, X^*) = \frac{1}{X^*}$ and want to show that, this is the Dulac function by computing the divergence of

$$\frac{1}{X^*}\left(\frac{1}{T^*}(S_0^* - S^*) - X^*\frac{S^*}{1+S^*}, \frac{1}{T^*}(-X^*) + X^*\frac{S^*}{1+S^*} - K_d^* X^*\right) = \left(\frac{1}{T^*}\frac{1}{X^*}(S_0^* - S^*) - \frac{S^*}{1+S^*}, \frac{-1}{T^*} + \frac{S^*}{1+S^*} - K_d^*\right)$$

Firstly, compute the derivative with respect tos^* of the first component, we get

$$\frac{-1}{T^*}\frac{1}{X^*} - \frac{1}{(1+S^*)^2}$$

Secondly, compute the derivative with respect to $X^*$of the second component and we get 0. Then the divergence is equal to the derivative with respect to X^* of the second component and we get 0. Then the divergence is equal to

$$\frac{-1}{T^*}\frac{1}{X^*} - \frac{1}{(1+S^*)}$$

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Both terms$\rho = \frac{1}{X^*}$therefore, ￼it is a Dulac function. This means there are no periodic solutions in the region R. No periodic solution in the region R.

Steady-state solution

The steady-state solution for the system requires that$\frac{dX^*}{dt^*} = 0$ means that

$$X^*\left(\frac{-1}{T^*} + \frac{S^*}{1+S^*} - K_d^*\right) = 0$$

Therefore,$X^* = 0$ orthat$\frac{S^*}{1+S^*} = \frac{1}{T^*} + K_d^*$

In the first case, we obtain from the requirement that ￼. Therefore, $S^* = S_0^*$this produces the "washout solutio$\frac{dS^*}{dt^*} = 0$ ￼. In the second case we solve for S^* and we find that erefore, $S^* = S_0^*$ which produce the "washout solution" $(S^*, X^*) = (S_0^*, 0)$. In the second case we solve for S^* and we find that

$$\frac{S^*}{1+S^*} = \frac{1+K_d^*T^*}{T^*}$$
$$1 + K_d^*T^* = (T^* - K_d^*T^* - 1)S^*$$
$$1 + K_d^*T^* = \left((1-K_d^*)T^* - 1\right)S^*$$

$S^* = S^{\hat{}*} = \frac{1+T^*K_d^*}{(1-K_d^*)T^*-1}$

Now solve forwhat X^*we find

$$0 = \frac{1}{T^*}(S_0^* - S^*) - \frac{X^*S^*}{1+S^*}$$
$$X^* = \frac{1}{T^*}(S_0^* - S^*)\frac{1+S^*}{S^*}$$

But $\frac{1+S^*}{S^*} = \frac{T^*}{1+K_d^*T^*}$therefore,

$$X^* = \frac{1}{T^*}(S_0^* - S^{\hat{}*}) \Big/ \left(\frac{1}{T^*} + K_d^*\right)$$

Which produce the "nowashout solution"

$$(S^*, X^*) = \left(S^{\hat{}*}, \frac{S_0^* - S^{\hat{}*}}{1+K_d^*T^*}\right)$$

Thecondition requirement for a physically meaningful steady state solution

The steady state solution to be physically meaningful requires that $S^* > 0$and $X^* > 0$

The first requirement is thatS^* is positive, which is the case exactly when the denominator

$$(1-K_d^*)T^* - 1 > 0$$

This leads to the requirement

$$T^* > \frac{1}{1-K_d^*} \qquad (F1)$$

For this to be physically meaningful, we must require that $1 - K_d^* > 0$and $K_d^* > 0$

$$0 < K_d^* < 1 \qquad (F_2)$$

The second requirement is that $X^* > 0$ which is equivalent to $S^* < S_0^*$ or

$$\frac{1+K_d^*T^*}{(1-K_d^*)T^* - 1} < S_0^*$$

Solving this inequality forT^*yields

$$T^* > \frac{1+S_0^*}{(1-K_d^*)S_0^* - K_d^*} \qquad (F_3)$$

For the numerator of this equation to be positive, we must require that

$$S_0^* > \frac{K_d^*}{1-K_d^*} (F_4)$$

If the condition $(F_4)$is true,(F_4)en K_d^*it must be less than,1 and so(F_2) is satisfiedK_d^* must be less than 1 and so(F_2) is satisfied

If the condition (F_3) is true, then (F_3) is true then

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$$T^* > \frac{1+S_0^*}{(1-K_d^*)S_0^*}$$

Because $k_d^* > 0$ .However, then

$$T^* > \frac{S_0^*}{(1-K_d^*)S_0^*} = \frac{1}{1-K_d^*}$$

And therefore, $(F_1)$is satisfied. Therefore, conditions forthe no-washout branch to be physically meaningful are,

$$T^* > \frac{1+S_0^*}{(1-K_d^*)S_0^* - K_d^*} \qquad (F_3)$$

$$S_0^* > \frac{K_d^*}{1-K_d^*} (F_4)$$

we have

$$S^* = \frac{1+K_d^*T^*}{(1-K_d^*)T^*-1}$$

If we take the derivative with respect to,T^*we obtain

$$\frac{dS^*}{dT^*} = \frac{-1}{\left((1-K_d^*)T^*-1\right)^2}$$

Which is negative. This means physically that if we increase residence time, the substrate concentration in equilibrium will decrease.

Maximum microorganism concentration as a function of T^*

For the no-washout state, we obtain explicitly:

$$X^* = \frac{S_0^*}{1+K_d^*T^*} = \frac{1}{(1-K_d^*)T^*-1}$$

Byderiving this with respect toT^* we get

$$\frac{-K_d^*S_0^*}{(1+K_d^*T^*)^2} + \frac{(1-K_d^*)}{\left((1-K_d^*)T^*-1\right)^2}$$

Setting this derivative equal to 0 gives

$$(1-K_d^*)K_d^*((S_0^*(1-K_d^*)-K_d^*)T^{*^2} - 2(1-K_d^*)(K_d^*S_0^*+K_d^*)T^* + K_d^*S_0^* + K_d^* + 1 = 0$$

It is a quadratic equation, for $T^*$and it has two positive roots.

Stability

Jacobian Matrix

The stability of the steady state solution for the system (1.5) and (1.6) are determined by the eigenvalues of the Jacobian matrices evaluated at the steady state solution.

The Jacobian matrix is defined by

$$J = \begin{bmatrix} f_{S^*}(S^*,X^*) & f_{X^*}(S^*,X^*) \\ g_{S^*}(S^*,X^*) & g_{X^*}(S^*,X^*) \end{bmatrix}$$

Local Stability

For the washout branch, the Jacobian matrix becomes

$$\begin{bmatrix} \frac{-1}{T^*} & \frac{-S^*}{1+S_0^*} \\ 0 & \frac{-1}{T^*} + \frac{S_0^*}{1+S_0^*} - K_d^* \end{bmatrix}$$

The eigenvalues are

$\lambda_1 = \frac{-1}{T^*}, \lambda_2 = \frac{-1}{T^*} + \frac{S_0^*}{1+S_0^*} - K_d^*$

Local stability requires that both eigenvalues are negative. The first is negative, and the second is negative if

$$\frac{S_0^*}{1+S_0^*} < K_d^* + \frac{1}{T^*}$$

We observe then that

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$$\frac{S_0^*}{1+S_0^*} < K_d^* \rightarrow \frac{S_0^*}{1+S_0^*} < K_d^* + \frac{1}{T^*}$$

When $K_d^* < \frac{S_0^*}{1+S_0^*}$stability requires, then that T^* be small enough so thatll enough so that

$$\frac{S_0^*}{1+S_0^*} < K_d^* + \frac{1}{T^*}$$

For nowashout branch, the Jacobian matrix given by

$$\begin{bmatrix} \frac{-1}{T^*} - \frac{X^*}{(1+S^*)^2} & \frac{-S^*}{1+S^*} \\ \frac{X^*}{(1+S^*)^2} & 0 \end{bmatrix}$$

and$detJ = \frac{X^*S^*}{(1+S^*)^3}$

When the solution is physically meaningful, the determinant is greater that0.

$$trJ = \frac{-1}{T^*} - \frac{X^*}{(1+S^*)^2}$$

And it is negative if X^* and $S^*$are positive. Therefore, if the washout solution is physically meaningful. Then,it is locally stable.

Transcriticalbifurcations

Atranscritical bifurcation occurs at the parameter

$$T_{tr}^* = \frac{1+S_0^*}{(1-K_d^*)S_0^* - K_d^*}$$

The value of the resident time at the transcritical bifurcation represents the minimum resident time at which the treatment process fails. At lower residence times, microorganisms are removed from the system at a rate greater than their maximum growth rate, resulting in process failure. At residence time (lower or higher) than the transcritical value, the washout (nowashout) solution is the only stable solution. We know that the bioreactor model has two solution branches. These are the washout branches,which are stable for

$$T_{tr}^* < \frac{1+S_0^*}{(1-K_d^*)S_0^* - K_d^*}$$

And the no-washout branch,which is stable for

$$T_{tr}^* > \frac{1+S_0^*}{(1-K_d^*)S_0^* - K_d^*}$$

Global Stability

When the washout solution is locally stable

We first show that when the washoutsolution is locally stable, then it is globally stable. First, recall that$R$ is positively invariant and exponentially attracting. It therefore suffices to show that the solutions start in R tend to the washout solution.

Local stability means that both eigenvalues are negative and,that:

$$\frac{S_0^*}{1+S_0^*} < k_d^* + \frac{1}{T^*}$$

Since the function $\frac{S^*}{1+S^*}$ on the interval $[0, S_0^*]$ has its maximum at S_0^* we conclude that for all S^* in this interval

$$\frac{S^*}{1+S^*} < K_d^* + \frac{1}{T^*}$$

or equivalently,

$$-\frac{1}{T^*} + \frac{S^*}{1+S^*} - K_d^* < 0$$

Since the interval$[0, S_0^*]$ is compact, there exists an $\epsilon > 0$so that

$$-\frac{1}{T^*} + \frac{S^*}{1+S^*} - K_d^* < -\epsilon$$

If we now consider the evolution equation for X^* in the region R

$$\frac{dX^*}{dt^*} = \left(-\frac{1}{T^*} + \frac{S^*}{1+S^*} - K_d^*\right) X^*$$

We see that

$$\frac{dX^*}{dt^*} < -\epsilon X^*$$

And we conclude th$R$in R the solution tends to 0 at an exponential rate.tend to 0 at an exponential rate.
If we now consider the equation for S^* we conclude that$S^*(t^*)$then tends toS_0^*at an exponential rate.
Therefore, all solutions that start in Rtend to the washout solution, and we conclude that the washout solution is globally stable.We tendto the washout solution, and we conclude that the washout solution is globally stable.

When the washout solution has a zero eigenvalue

We show that in this case, the washout solution is also globally stable. In this case, we have that

$$\frac{S_0^*}{1+S_0^*} = K_d^* + \frac{1}{T^*}$$

Since the function $\frac{S^*}{1+S^*}$on the interval $[0, S_0^*]$ has its maximum atS_0^*we conclude that for all $S^* < S_0^*$in this interval

$$\frac{S^*}{1+S^*} < {K^*}_d + \frac{1}{T^*}$$

Now consider the equation for X^*again:

$$\frac{dX^*}{dt^*} = \left(-\frac{1}{T^*} + \frac{S^*}{1+S^*} - K_d^*\right) X^*$$

Therefore,the coefficient in front of $X^*$is negative at all points in R. This means that $X^*$is monotonically decreasing in time and at an exponential rate. Therefore, for any solution$(S^*(t^*), X^*(t^*))$tstartstart in RtheX^*coordinate must become arbitrarilysmall. But then, if we consider the equation for S^*:

$$\frac{dS^*}{dt^*} = \frac{1}{T^*}(S_0^* - S^*) - \frac{X^* S^*}{1+S^*}$$

the S^*values tend totherefore as tends to the washout solutionS_0^*ce $R$is exponentially attracting and positively invariant, we conclude that this is the case for a physically meaningful solution and therefore the washout solution is still globally stable.therefore as $t \to \infty$ $(S^*(t^*), X^*(t^*))$tend to the washout solution. Since $R$is exponentially attracting and positively invariant we conclude that this is the case for a physically meaningful solution and therefore the washout solution is still globally stable.

Lyapunov function for the no-washout solution

Letdenote the no-washout solution, and we suppose that it is physical:both coordinates are positive. The Lyapunovfunct $(S^{*\hat{}}, X^{*\hat{}})$ was found in [7]. Definephysical:both coordinates are positive. The Lyapunove function was found in [7]. Define

$$V(S^*, X^*) = \int_{S^{\hat{}*}}^{S^*} \frac{s^* - S^{\hat{}*}}{s^*} ds^* + (1 + S^{\hat{}*}) \int_{X^{\hat{}*}}^{X^*} \frac{x^* - X^{\hat{}*}}{x^*} dx^*$$

This function is well defined when both $S^*$and X^* are greater than zero

$$\frac{\partial V}{\partial S^*} = \frac{S^* - S^{\hat{}*}}{S^*} \quad (1.7)$$

$$\frac{\partial V}{\partial X^*} = (1 + S^{\hat{}*}) \frac{x^* - X^{\hat{}*}}{x^*} \quad (1.8)$$

By computation, we show that for a solution not at the washout solution

$$\frac{d}{dt^*} V(S^*(t^*), X^*(t^*)) < 0$$

This then implies V is decreasing and therefore that solution tends to the no-washout solution.
In order to make the following computation easier to see, we simplify some of the notation in the equation, and rewrite the differential equation using new quantities $D = \frac{1}{T^*}$and $d = \frac{1}{T^*} + K_d^*$as:

$$\frac{dS^*}{dt^*} = D(S_0^* - S^*) - \frac{X^* S^*}{1+S^*}$$

$$\frac{dX^*}{dt^*} = X^* \left(\frac{S^*}{1+S^*} - d\right)$$

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For later use, we evaluate the coefficient

$$\left(\frac{S^*}{1+S^*}-d\right)=\left(\frac{S^*}{1+S^*}-\frac{S^{\wedge *}}{1+S^{\wedge *}}\right)=\frac{S^*-S^{\wedge *}}{(1+S^{\wedge *})(1+S^*)}$$

We obtain

$$\frac{d}{dt^*}V\big(S^*(t^*),X^*(t^*)\big)=\frac{\partial V}{\partial S^*}\frac{dS^*}{dt^*}+\frac{\partial V}{\partial X^*}\frac{dX^*}{dt^*}$$
$$=\left(\frac{S^*-S^{\wedge *}}{S^*}\right)\left(D(S_0^*-S^*)-\frac{X^*S^*}{1+S^*}+(1+S^{\wedge *})\frac{X^*-X^{\wedge *}}{X^*}X^*\left(\frac{S^*}{1+S^*}-d\right)\right.$$

Now the first term is equal to:

$$(S^*-S^{\wedge *})(D\frac{{S_0}^*-S^*}{S^*})\_(S^*-S^{\wedge *})\frac{X^*}{1+S^*}$$

The second term is equal to

$$(1+S^{\wedge *})\big(X-X^{\wedge *}\big)\left(\frac{S^*}{1+S^*}-d\right)=(X^*-X^{\wedge *})\frac{S^*-S^{\wedge *}}{1+S^*}$$

Therefore, if we add the two terms together, the terms involving X^* cancel and the summation becomes (note we can factor out):

$$(S^*-S^{\wedge *})(D\frac{S_0^*-S^*}{S^*}-\frac{X^{\wedge *}}{1+S^*})$$

Since $X^{\wedge *}=(S_0^*-S^{\wedge *})\frac{D}{d}$the last expression is equal to (note we can also factor out a D)

$$(S^*-S^{\wedge *})D(\frac{S_0^*-S^*}{S^*}-\frac{S_0^*-S^{\wedge *}}{d(1+S^*)})$$

Now consider the second expression in parentheses

$$\frac{S_0^*-S^*}{S^*}-\frac{S_0^*-S^{\wedge *}}{d(1+S^*)}=\frac{S_0^*-S^*}{S^*}-\frac{(S_0^*-S^{\wedge *})(1+S^{\wedge *})}{S^{\wedge *}(1+S^*)}$$

When we make a common denominator in this last expression and simplify it becomes equal to a much simpler expression, namely:

$$-(S^*-S^{\wedge *})\frac{S_0^*+S^{\wedge *}S^*}{S^{\wedge *}S^*(1+S^*)}$$

Combining these, we obtain:

$$\frac{dV(S^*(t^*),X^*(t^*))}{dt^*}=-(S^*-S^{\wedge *})^2D\frac{S_0^*+S^{\wedge *}S^*}{S^{\wedge *}S^*(1+S^*)}$$

Which is clearly non-positive.

Therefore, $V$is then a Lyapunov function for the system and proves that the no-washout solution is globally stable when it is physical.

Steady-state Diagrams

With $K_s=350mgl^{-1}$, $S_0=250mgl^{-1}$,α=0.5 , $K_d=0.05\,day^{-1}$and $\mu_m=3\,day^{-1}$,we obtain as dimensionless parameters:

$S_0^*=\frac{S_0}{K_s}=0.7143$, $K_d^*=\frac{K_d}{\mu_m}=0.017$

These choices for the parameter yield$T_{CT}^*=2.500$ can be obtained from call T^*These choices for the parameter yield$T_{CT}^*=2.500$and can be obtained from call T^*

Exact steady-state diagrams

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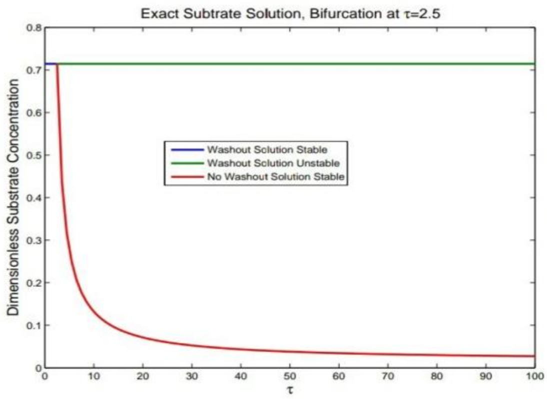
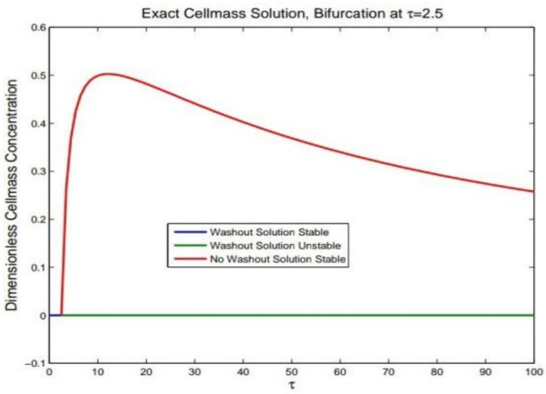
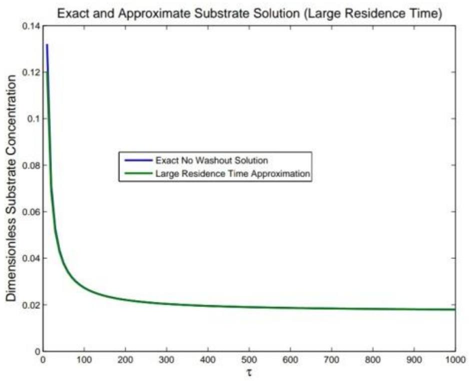


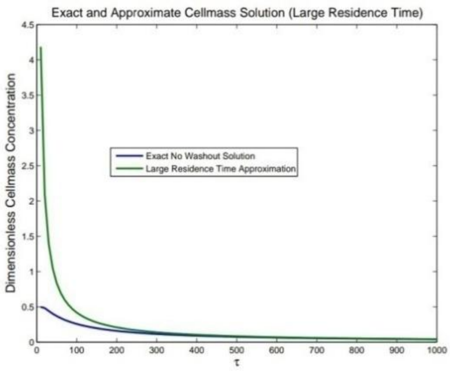


The exact solutions for the dimensionless substrate concentration and cell mass are in the two figures above. The bifurcation that occurs is that the moment the no-washout solution becomesphysical,it also becomes stable, while the no-washout solution then becomes unstable.

Steady-state diagrams (large residence time approximation)

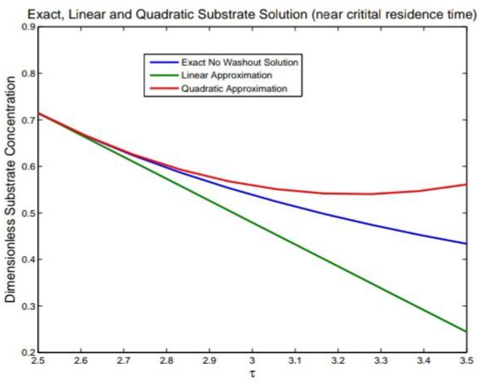
The comparisons between the exact solution and the large residence time approximation are shown in the next two graphs with respect to residence time. It is noteworthy that the approximation for the substrate concentration remains good over the entireinterval, while the cell mass concentration becomes relatively poor.

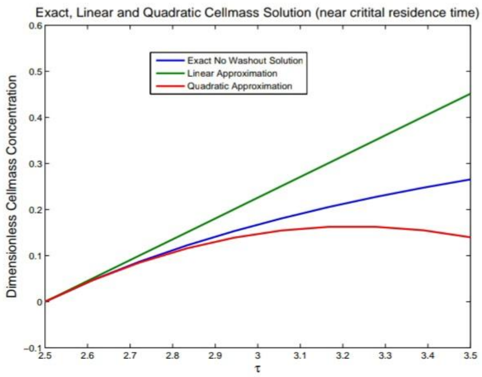




Steady-state diagrams(residence time slightly larger than critical value approximation)

We produce the plots for the steady-state diagrams for residence time slightly larger than the critical value. And show the exact plots and those obtained from the second-order Taylor series (linear approximation) and those obtained from the third-order Taylor series (quadratic approximation).





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Conclusion

To sum up, we conclude that, for the success of the wastewatertreatment process,it was represented as a system of two nonlinear equations. There are some conditions that should be taken into consideration. If $K_d \geq \frac{S_0^*}{1+S_0^*}$then the solution convergesto $(S_0^*, 0)$. In practical applications, we want to avoid this solution. This makes biological sense because, in dimensionalunits, this means that $K_d^* \geq \mu_m$,meaning that the decay is faster than the specific growth while in residence. S$X_0^* = 0$ no material enters the bioreactor over time microorganism decreases. In physically meaningful solutions, if we increase the residence time (the time for the reaction to take place) the substrate concentration in equilibrium will decrease over the time. $K_d \geq \frac{S_0^*}{1+S_0^*}$ maximizemicroorganism concentration, weneed tomaXX^* as a function ofT^*. The local st$(S_0^*, 0)$ity for washout branch to be stable requirest^* be small enough to get$\frac{-1}{T^*} + \frac{S_0^*}{1+S_0^*} - K_0^* < 0$. If the washout solutionsarephysically meaningful (i.e$S^* > 0$ $X^* > 0$ then it is locally stable. A transcritical bifurcation occurs at the parameter unite$K_d^* \geq \mu_m$ meaning$X_0^* = 0$ no material enters the bioreactor over themicroorganism decreases. In physically meaningful solutions, if we increase the resident time (the time for the reaction to take place) the substrate concentration in equilibrium will decrease overtime. To maximum microorganism concentration, we need to X^* T^*. The local stability for washout branch to be t^* $\frac{-1}{T^*} + \frac{S_0^*}{1+S_0^*} - K_0^* < 0$. If the washout solutions is physically meaningful (i.e$S^* > 0$ $X^* > 0$ then it is locally stable. A transcritical bifurcation occurs at the parameter

$$T_{tr}^* = \frac{1 + S_0^*}{(1 - K_0^*)S_0^* - K_d^*}$$

The Lyapunov function for the system proves that theno washout solution is globally stable when it is physical. As well asThe washout solution is globally stable.

Future work

In this paper, we studied how we can increase the number of microorganisms inside the bioreactor, through the consumption of the substrate flowing in the bioreactor. This means, the substrate is only the source of the food for microorganisms. However, what happens if this substrate is toxic or non-competitive product inhibition. How can that affect the reaction and the system form.We willbe studying in the future.

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